Leading-order hadronic vacuum polarization contribution to the anomalous magnetic moment of the muon in Lattice QCD with four flavors of quarks at their physical mass

Rehan Malak
Advisor : Laurent Lellouch

CPT,CNRS Aix-Marseille Univ., Univ. de Toulon/MDLS,CEA Saclay

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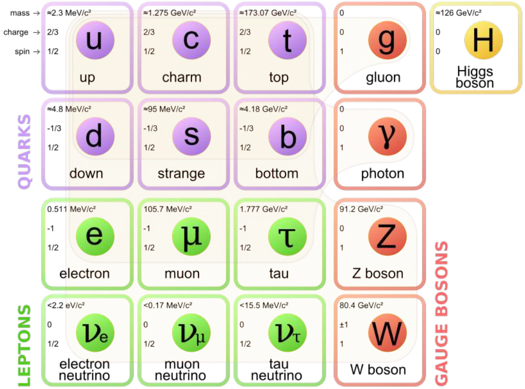
1) Motivation : amu as a precision test of the Standard Model and the amuhvplo challenge

The place of the muon in the standard model (SM)



Muon :

- lepton $\Rightarrow$ no strong interaction
- electric charge $\Rightarrow$ electromagnetic interaction
- $+$ intrinsic angular momentum

$\Rightarrow$ acts as a magnet in a magnetic field

(Anomalous) magnetic moments of charged leptons

Lepton magnetic moment :
$$\vec{M} = g_\ell \frac{e}{2m_\ell} \vec{S}$$

- Dirac (1928)

$$g_\ell = 2$$

- $\ell = e$, Kusch & Foley (1948) experiments

$$g_e = 2.001\,18(3)$$

Lepton **anomalous** magnetic moment :
$$a_\ell := \frac{g_\ell - 2}{2}$$

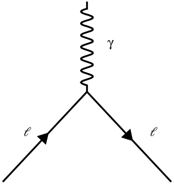
- Schwinger (1948) : first quantum correction

$$a_\ell = \frac{\alpha}{2\pi} \approx 0.001\,161$$

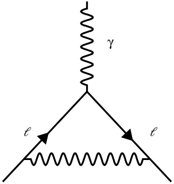
$\Rightarrow$ Huge success for quantum electrodynamics (QED)

Modern view

Dirac value = Tree-level QED :



Schwinger value = 1-loop QED :

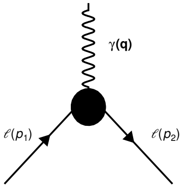


Experiment/Theory interplay

Increasing experimental precision $\Leftrightarrow$ theory accounts for :

- for smaller and smaller quantum corrections
- more and more interactions

In a theory which respects parity and time reversal invariance :



$$= \quad -ie\Big[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m_\mu}F_2(q^2)\Big]$$

$$\rightarrow F_2(0) = a_\ell$$

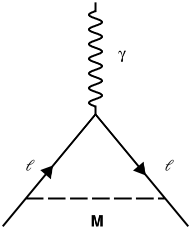
Why the muon?

$$m_e : m_\mu : m_\tau \simeq 0.511 : 105. : 1777.\ \text{MeV}$$

$$\tau_e : \tau_\mu : \tau_\tau \simeq \text{stable} : 2. \times 10^{-6} : 3. \times 10^{-13}\ \text{s}$$

Berestetskii (1956)

$$a_\ell \propto \frac{m_\ell^2}{M^2}$$

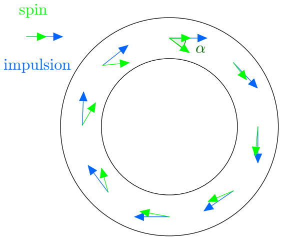


Muon :

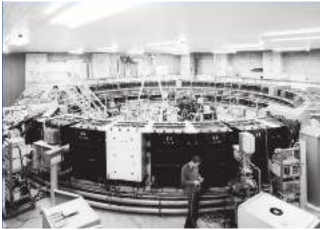
- $(m_\mu/m_e)^2 \sim 4 \cdot 10^4 \times$ more sensitive to new physics than e
- lives long enough to study a_μ, unlike τ

a_μ measures Larmor precession :

$$\omega_{\text{Larmor}} = \omega_{\text{Spin}} - \omega_{\text{Cyclotron}} = a_\mu \frac{eB}{m_\mu}$$



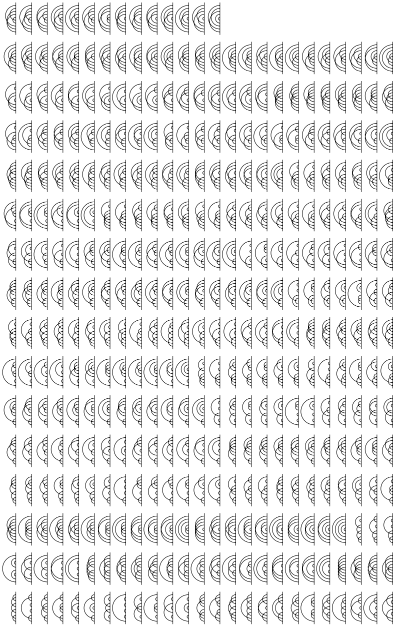
Experiment	Year	μ^+/μ^-	$a_\mu \times 10^{10}$ (δa_μ)	Precision [ppm]
CERN I	1961	μ^+	11 450 000(220000)	4300
CERN II	1962-1968	μ^+	11 661 600(3100)	270
CERN III	1974-1976	μ^+	11 659 100(110)	10
CERN III	1975-1976	μ^-	11 659 360(120)	10
BNL	1997	μ^+	11 659 251(150)	13
BNL	1998	μ^+	11 659 191(59)	5
BNL	1999	μ^+	11 659 202(15)	1.3
BNL	2000	μ^+	11 659 204(9)	0.73
BNL	2001	μ^-	11 659 214(9)	0.72
BNL	2008	Average	11 659 208.0(6.3)	0.54
BNL	2012	New avg.	11 659 209.1(6.3)	0.54



Quantum electrodynamics

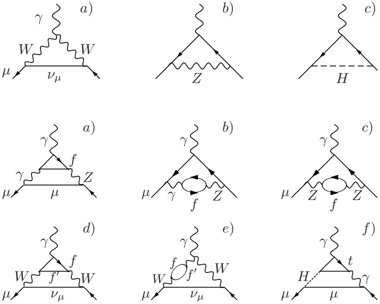
- Gives largest contribution to a_μ
- Calculated to $O(\alpha^5)$ [Aoyama et al, 2006-2015]
- 12 672 diagrams
- $O(\alpha^5)$ correction to $a_\mu = 0.50938(70) \times 10^{-10}$

$\Rightarrow$ no need to go further



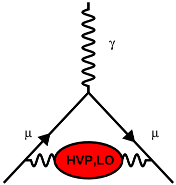
Weak interaction

- Contribution to a_μ only became relevant for BNL E821
- Calculated to 2 loops [Czarnecki et al 1996, Knecht et al 2002, Gnendinger et al 2013]



$$a_\mu^{(2)\mathrm{EW}} = 19.482(2) \times 10^{-10}$$
$$a_\mu^{(4)\mathrm{EW}} = -4.12(10) \times 10^{-10}$$

$\Rightarrow$ no need to go further



- $O(\alpha^2)$ at LO but mainly nonperturbative in QCD
- usually obtained from :

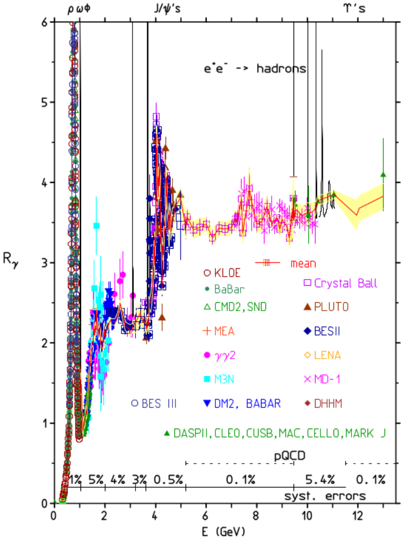
$$e^+e^- \to \text{hadrons}$$

- uses dispersion relation

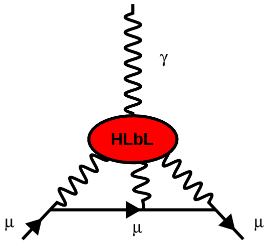
⇒ recent estimate (e.g. [Jegerlehner 2015])

$$a_\mu^{\text{QCD,LO}} = 687.0(4.2) \times 10^{-10}$$

- NLO & NNLO also determined (e.g. [Kurz et al 2014])



Hadronic Light-by-light (HLbyL)



- $O(\alpha^3)$ & cannot be fully obtained via experiment
- estimate = Glasgow consensus [Prades et al, 2009]

$$a_\mu^{\mathrm{QCD,LbL}} = 10.5(2.6) \times 10^{-10}$$

- Recent Lattice estimate [RBC 2016]

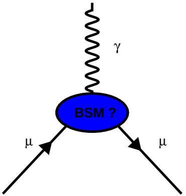
$$a_\mu^{\mathrm{QCD,LbL}} = 5.35(1.35) \times 10^{-10} \quad \text{(only statistical)}$$

Precision test of the SM and evidence for new physics?

			$a_\mu \cdot 10^{10}$	$\delta a_\mu \cdot 10^{10}$
	QED	Aoyama '15	11 658 471.884 6	0.003 7
	EW	Gnendinger '13	15.36	0.1
QCD	QCD HVP,LO	Jegerlehner '16	687.0	4.2
	QCD HVP,NLO	Kurz '14	−9.934	0.091
	QCD HVP,NNLO	Kurz'14	1.226	0.012
	QCD LbL	Prades '09	10.5	2.6
Theory			11 659 176.1	5.0
Experiment		E821	11 659 209.1	6.3
Deviation			33.0	8.0

- 4.1σ discrepancy w/ SM $\rightarrow$ new physics?
- new E989 experiment $\Rightarrow \delta a_\mu^{\mathrm{EXP}} \rightarrow \delta a_\mu^{\mathrm{EXP}}/4$ begins in April 2017
- theory precision on HVP and HLbyL has to follow experiment
- crosscheck needed for the largest source of error $a_\mu^{\mathrm{HVP,LO}}$

$\Rightarrow$ NP QCD at 0.2% precision is huge challenge



2) Lattice Quantum Chromodynamics (LQCD)

What is quantum chromodynamics (QCD)?

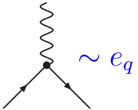
Fundamental theory of the strong force felt by quarks and gluons

Generalization of QED with only inputs, g and quark masses m_q:

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{2g^2}\,\mathrm{tr}\left[F_{\mu\nu}F_{\mu\nu}\right] + \sum_{q=u,d,s,c,b,t} \bar{\psi}_q \left[\gamma_\mu(\partial_\mu + A_\mu) + m_q\right]\psi_q$$

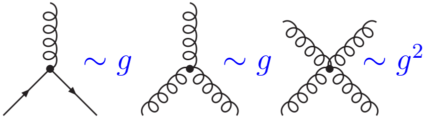
QED

q and γ interact through electric charge: $e \sim \sqrt{\alpha}$



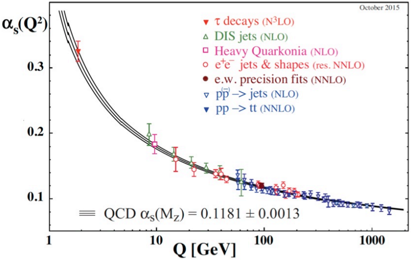
QCD

q and g interact through color charge: $g \sim \sqrt{\alpha_s}$



Asymptotic freedom and infrared slavery

Asymptotic freedom: interaction between quarks & gluons weakens as their relative momenta increase [Gross, Wilczek, Politzer '73]



[PDG '15]

Infrared slavery: quarks & gluons are inextricably **confined** within **hadrons**

Difficult to describe mathematically: the theory must produce a ``sticky mess'' of quarks & gluons

$\rightarrow$ numerical simulations in lattice QCD

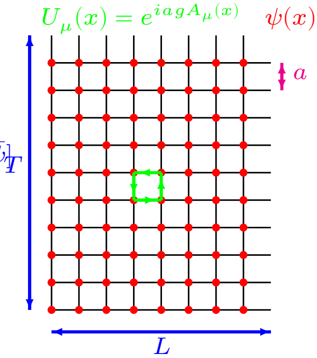
What is lattice QCD (LQCD)?

To describe ordinary matter, QCD requires ≥ 104 numbers at every point of spacetime
$\rightarrow \infty$ number of numbers in our continuous spacetime
$\rightarrow$ must temporarily ``simplify'' the theory to be able to calculate (regularization)
$\Rightarrow$ Lattice gauge theory $\longrightarrow$ mathematically sound definition of NP QCD:

- UV (& IR) cutoff $\rightarrow$ well defined path integral in Euclidean spacetime:

$$\langle O \rangle = \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi \, e^{-S_G - \int \bar{\psi} D[M] \psi} \, O[U, \psi, \bar{\psi}]$$
$$= \int \mathcal{D}U \, e^{-S_G} \det(D[M]) \, O[U]_{\text{Wick}}$$

- $\mathcal{D}U e^{-S_G} \det(D[M]) \geq 0$ & finite # of dofs $\rightarrow$ evaluate numerically using stochastic methods



LQCD is QCD when $m_q \rightarrow m_q^{\text{ph}}$, $a \rightarrow 0$ (after renormalization), $L \rightarrow \infty$ (and stats $\rightarrow \infty$)

HUGE conceptual and numerical ($\sim 10^9$ dofs) challenge

Wilson vs staggered fermions

We have used two different fermion discretizations

Wilson

add Wilson term :

$$\begin{aligned} \mathcal{S}_D{}^{\psi} &= a^4 \sum_{x\in\Lambda} \sum_{\mu} \bar{\psi}(x)\gamma_\mu \frac{\nabla^*_\mu + \nabla_\mu}{2} \psi(x) \\ &+ a^4 \sum_{x\in\Lambda} m_\psi \bar{\psi}(x)\psi(x) \\ &- \frac{a^5}{2} \sum_{x\in\Lambda} \sum_{\mu} \bar{\psi}(x) \nabla^*_\mu \nabla_\mu \psi(x) \\ &\underset{a\to 0}{\to} \bar{\psi}(\partial\!\!\!/ + m_\psi)\psi + i\bar{\psi} A\!\!\!/ \psi \end{aligned}$$

Advantages :

- Has continuum flavor symmetry

Drawbacks :

- Hard breaking of chiral symmetry
  $\rightarrow$ additive mass renormalization
- Fermion matrix badly conditionned

$\Rightarrow$ propagators and HMC expensive for small m_ψ

$\Rightarrow$ too expensive today for precision required

Staggered

mix space-time and spinor d.o.f. with sign function :

$$\eta_1(n) = 1 \; , \; \eta_2(n) = (-1)^{n1}$$
$$\eta_3(n) = (-1)^{n_1+n_2} \; , \; \eta_4(n) = (-1)^{n_1+n_2+n_3}$$

and keep 1 d.o.f. χ in the diagonalized action :

$$\begin{aligned} \mathcal{S}_D{}^{\chi} &= a^4 \sum_{x\in\Lambda} \sum_{\mu} \bar{\chi}(x)\eta_\mu(x) \frac{\nabla^*_\mu + \nabla_\mu}{2} \chi(x) \\ &+ a^4 \sum_{x\in\Lambda} m_\chi \bar{\chi}(x)\chi(x) \end{aligned}$$

Drawbacks :

- Flavor symmetry is broken

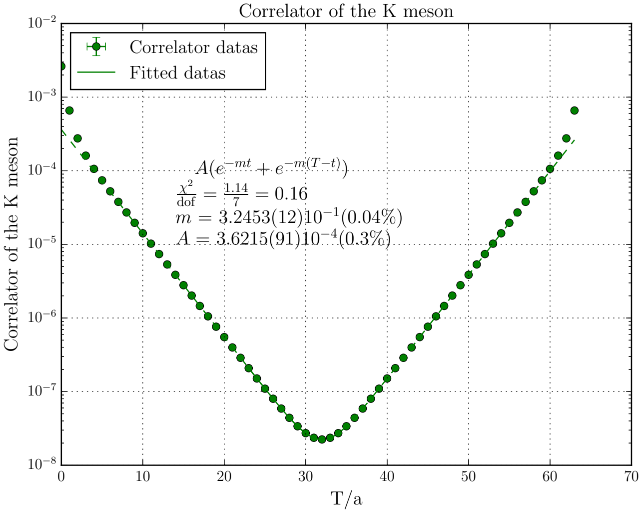
$\Rightarrow$ states are mixtures of different ``tastes''

Advantages :

- Have a $U(1)$ chiral symmetry
- Fermion matrix 4x smaller, better conditionned
- State-of-the-art codes $\Rightarrow$ physical parameters !

Time-momentum propagator of kaon at rest

$$C(t) \equiv \frac{1}{(L/a)^3} \sum_{\vec{x}} \langle [\bar{s}\gamma_5 d](x)[\bar{d}\gamma_5 s](0)\rangle \overset{a \ll t \ll T}{\longrightarrow} \frac{|\langle 0|\bar{s}\gamma_5 d|K^0(\vec{0})\rangle|}{M_K} e^{-M_K \frac{T}{2}} \cosh\left[M_K\left(\frac{T}{2} - t\right)\right]$$



3) Computing amuhvplo in LQCD: finite-volume challenges (Wilson fermions)

Hadron vacuum polarization

Electromagnetic current (quark contribution) :

$$J_\mu^{\rm em} = \frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d - \frac{1}{3}\bar{s}\gamma^\mu s + \frac{2}{3}\bar{c}\gamma^\mu c[-\frac{1}{3}\bar{b}\gamma^\mu b + \frac{2}{3}\bar{t}\gamma^\mu t]$$

is conserved :

$$J_\mu^{(f)} = \bar{f}\gamma_\mu f \ , \ \partial_\mu J_\mu^{(f)} = 0$$

Hadronic vacuum polarization tensor in the euclidean :

$$\Pi_{\mu\nu}^{(ff')}(Q) = \mathrm{i}\int \mathrm{d}^4x\; e^{iQ\cdot x}\langle J_\mu^{(f)}(x) J_\nu^{(f')}(0)\rangle$$

respects Ward identity :

$$Q_\mu \Pi_{\mu\nu}^{(ff')}(Q) = Q_\nu \Pi_{\mu\nu}^{(ff')}(Q) = 0$$

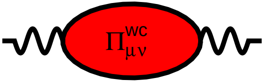
and decomposes as :

$$\Pi_{\mu\nu}^{(ff')}(Q) = (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2)\Pi^{(ff')}(Q^2)$$

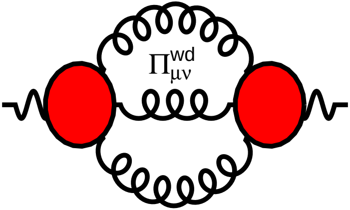
Flavor decomposition of HVP

$$\Pi_{\mu\nu} = \frac{4}{9}\Pi_{\mu\nu}^{(uu),\mathrm{wc}} + \frac{1}{9}\Pi_{\mu\nu}^{(dd),\mathrm{wc}} + \frac{1}{9}\Pi_{\mu\nu}^{(ss),\mathrm{wc}} + \frac{4}{9}\Pi_{\mu\nu}^{(cc),\mathrm{wc}} + \frac{1}{9}\Pi_{\mu\nu}^{(udsc),\mathrm{wd}}$$

Wick-connected contributions :



Wick-disconnected contributions :



⇒ this work focuses on the connected contributions

$a_\mu^{\mathrm{HVP,LO}}$ from the lattice

$a_\mu^{\mathrm{HVP,LO}}$ obtained from euclidean HVP computed on the lattice [Lautrup '69, Blum '02]

$$a_\mu^{\mathrm{HVP,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}Q^2 w(Q^2)\hat{\Pi}(Q^2)$$

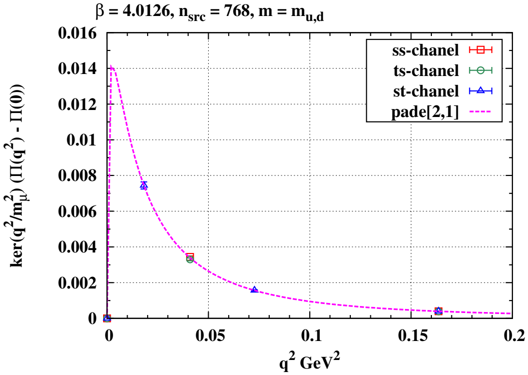
$$w(Q^2) = \frac{1}{4m_\mu^2} \frac{\left[(r+2)-\sqrt{r(r+4)}\right]^2}{\sqrt{r(r+4)}}\Bigg|_{r=\frac{Q^2}{m_\mu^2}}$$

$$\hat{\Pi}(Q^2) = \Pi(Q^2) - \Pi(0)$$

But :

- integrand peaks at kernel at $Q \sim m_\mu/2 \approx 0.053\,\mathrm{GeV} < 2\pi/L, T$
- $\Pi(0) = \frac{\Pi_{\mu\nu}(0)}{0} = \frac{0}{0}$ undefined!

$\Rightarrow$ need to control small $Q^2 \Leftrightarrow$ large x^2



HVP challenges: control small Q^2 $\leftrightarrow$ large distance

- In L^4, $Q_\mu \Pi_{\mu\nu}(Q) = 0$ does not imply $\Pi_{\mu\nu}(Q=0) = 0$

$$\Pi_{\mu\nu}(Q=0) = \int_\Omega d^4x \langle J_\mu(x) J_\nu(0)\rangle = \int_\Omega d^4x \partial_\rho [x_\mu \langle J_\rho(x) J_\nu(0)\rangle]$$

$$\int_{\partial\Omega} d^3x_\rho [x_\mu \langle J_\rho(x) J_\nu(0)\rangle] \propto L^4 \exp(-EL/2)$$

$\rightarrow$ distortion of $\Pi_{\mu\nu}(Q)/(Q_\mu Q_\nu - Q^2\delta_{\mu\nu})$ as $Q_\mu \rightarrow 0$

- Need $\Pi(0)$ to renormalize fine structure constant α: $\hat{\Pi}(Q^2) = \Pi(Q^2) - \Pi(0)$

$\Rightarrow$ test different approaches to determination of HVP in different volumes [BMWc '14]:

1. **Usual method:** compute $\Pi(Q^2) = \Pi_{\mu\nu}(Q)/(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2)$ and fit to rational function of Q^2 (i.e. Padé)
2. **Usual method w/ subtraction:** compute $\Pi(Q^2) = [\Pi_{\mu\nu}(Q) - \Pi_{\mu\nu}(0)]/(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2)$ and fit to rational function of Q^2 (i.e. Padé)
3. **Second derivative method:** obtain $\Pi(Q^2)$ directly (see below) and fit to rational function of Q^2 (i.e. Padé)

Second derivative method [BMWc '14]

Second set of methods considers Fourier derivatives of the polarization tensor : [BMWc '14]

$$\partial_\rho \partial_\sigma \Pi_{\mu\nu}(Q) = -a^4 \sum_x x_\rho x_\sigma \langle J_\mu(x) J_\nu(0) \rangle e^{iQ\cdot x} \ .$$

Appropriate choose of indices ρ, σ, μ, ν and four-momenta :

- $\rho = \sigma = \mu = \nu$ gives Adler function :

$$\mathcal{A}(Q^2) = Q^2 \frac{\partial \Pi(Q^2)}{\partial Q^2} = -\frac{1}{2} \partial_\mu \partial_\mu \Pi_{\mu\mu}(Q)|_{Q_\mu = 0}$$

- $\sigma = \mu \neq \rho = \nu$ gives :

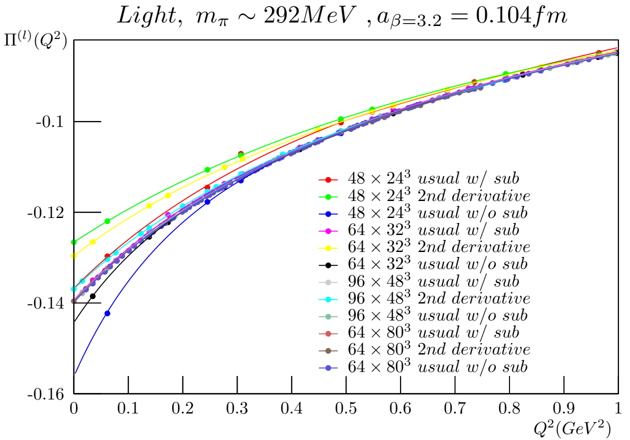
$$\Pi(Q^2) = \partial_\mu \partial_\nu \Pi_{\mu\nu}(Q)|_{Q_\mu = Q_\nu = 0}, \qquad \mu \neq \nu$$

- with $\sigma = \rho \neq \mu = \nu$, one can obtain :

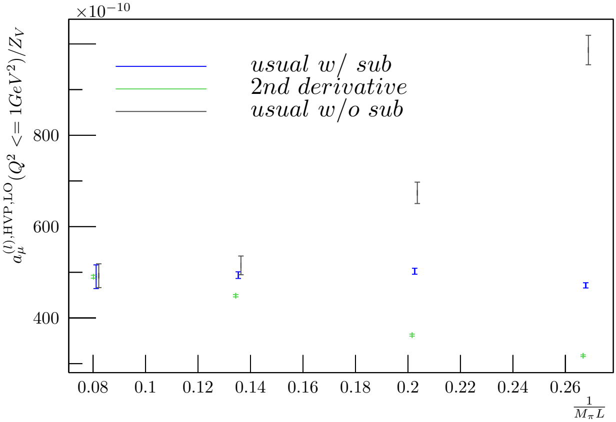
$$\Pi(0) = -\frac{1}{2} \partial_\sigma \partial_\sigma \Pi_{\mu\mu}(Q)\Big|_{Q=0}$$

Three methods in four volumes w/ Wilson fermions [BMWc '14]

T (fm)	L (fm)	m_π (MeV)	M_πT	M_πL
48a = 5.0	24a = 2.5	295.2(1.4)	7.5	3.7
64a = 6.7	32a = 3.3	292.6(7)	9.9	4.9
96a = 10.0	48a = 5.0	292.0(6)	14.8	7.4
64a = 6.7	80a = 8.3	292.1(3)	9.9	12.3



Finite-volume effects on $a_\mu^{\mathrm{HVP,LO}}$ [BMWc '14]



- Qualitatively similar results for s but smaller by up to a factor 8
- 2nd derivative method has smaller statistical errors but usual w/ subtraction has smaller FV corrections
  - $\Rightarrow$ **Must subtract** $\Pi_{\mu\nu}(0)$!
  - $\Rightarrow$ use usual method w/ subtraction
  - $\Rightarrow$ try getting $\Pi(0)$ renormalisation factor using 2nd derivative method

4) Results with Staggered fermions and physical parameters

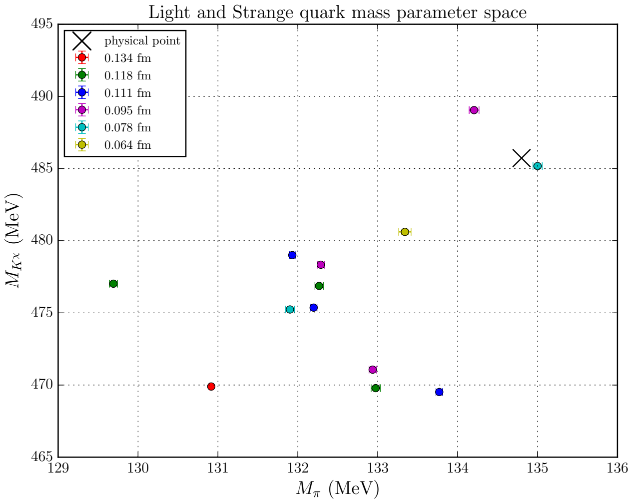
Staggered fermions at the physical mass point

13 Staggered simulations with $N_f = 2+1+1$ at 6 lattice spacings around physical mass point and on lattices of size $L \simeq 6\,\mathrm{fm}$ with $8.6 \leq T \leq 11.3\,\mathrm{fm}$

a (fm)	T (fm)	L (fm)	M_π (MeV)	$M_{K\chi}$ (MeV)	$M_\pi T$	$M_\pi L$
0.134	64a = 8.6	48a = 6.4	$1.30600(20)10^2$	$4.68764(50)10^2$	5.7	4.3
0.118	96a = 11.3	56a = 6.6	$1.32257(57)10^2$	$4.6725(12)10^2$	7.6	4.4
			$1.29873(50)10^2$	$4.7769(11)10^2$	7.5	4.3
			$1.32565(52)10^2$	$4.7794(11)10^2$	7.6	4.4
0.111	84a = 9.3	56a = 6.2	$1.31731(42)10^2$	$4.73684(94)10^2$	6.2	4.1
			$1.31920(36)10^2$	$4.78964(83)10^2$	6.2	4.2
			$1.32673(42)10^2$	$4.65663(88)10^2$	6.3	4.2
0.095	96a = 9.1	64a = 6.1	$1.34964(62)10^2$	$4.9182(14)10^2$	6.2	4.2
			$1.31787(44)10^2$	$4.76526(89)10^2$	6.1	4.1
			$1.31468(45)10^2$	$4.6586(11)10^2$	6.1	4.1
0.078	128a = 10.0	80a = 6.2	$1.31953(58)10^2$	$4.7542(12)10^2$	6.7	4.2
			$1.36299(58)10^2$	$4.8984(13)10^2$	6.9	4.3
0.064	144a = 9.2	96a = 6.1	$1.32973(76)10^2$	$4.7929(20)10^2$	6.2	4.1

Mass landscape of ensembles

$$M_\pi^2 \propto m_{ud} = \frac{m_u + m_d}{2} \qquad\qquad M_{K^\chi}^2 \propto m_s$$



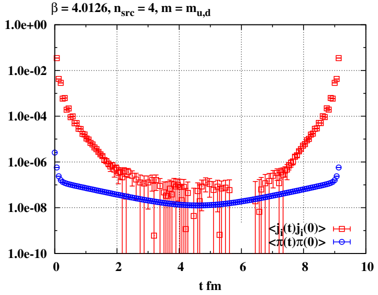
HVP challenges: light pions and statistics

- Physically light pions only being used very recently [BMWc '13 & in progress, HPQCD/MILC '15 & in progress, RBC/UKQCD in progress]

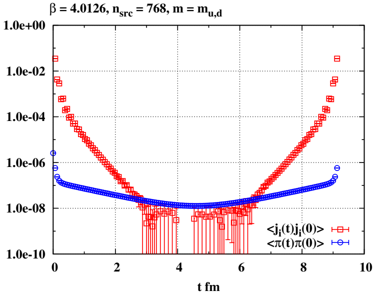
- Errors in $\langle\pi(t)\pi(0)\rangle$ & $\langle J_i^{ud}(t)J_i^{ud}(0)\rangle_{WC}$ as fn of t [BMWc '16]

m_{ud}, m_s, m_c physical, $a \simeq 0.064\,\text{fm}$, $L = 96a \simeq 6.1\,\text{fm}$, $T = 144a \simeq 9.2\,\text{fm}$

Good stats: 4×441 meas.



For $\delta_{\text{stat}} a_\mu^{\text{HVP,LO}} \sim 1\%$: 768×441 meas.



$\rightarrow$ required many algorithmic improvements

Three methods for determining $\hat{\Pi}(Q^2)$ at low Q^2

Investigate 3 methods for determining $\hat{\Pi}(Q^2)$ at low Q^2 from $\langle J_\mu^{(f)}(x) J_\mu^{(f')}(0)\rangle$, based on findings with Wilson fermions

1. **Traditional:** compute $\Pi(Q^2) = [\Pi_{\mu\nu}(Q) - \Pi_{\mu\nu}(0)]/\,(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2)$ vs Q^2, fit to low order rational function (Padé) and build $\hat{\Pi}(Q^2) = \Pi(Q^2) - \Pi(0)$ from extrapolated $\Pi(0)$
2. **Add 2nd moment:** same as above but subtract $\Pi(0)$ obtained from 2nd derivative method

$$\hat{\Pi}(Q^2) = \sum_{t,\vec{x}} \mathrm{Re}\left(\frac{\exp(iQt) - 1}{Q^2} + \frac{1}{2}t^2\right) \mathrm{Re}\langle J_\mu(t,\vec{x}) J_\mu(0)\rangle$$

3. **Taylor:** determine parameters of rational function (Padé) approximant from Taylor coefficients

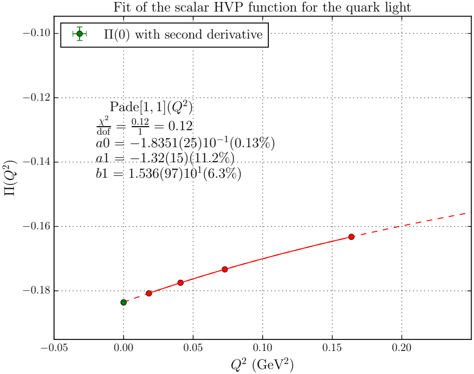
$$\Pi_j = \frac{\partial_Q^{2(j+1)} \Pi_{\mu\mu}\Big|_{Q=0}}{(2(j+1))!} = (-)^{n+1} \sum_x \frac{\hat{x}_\nu^{2n+2}}{(2n+2)!} \mathrm{Re}\langle J_\mu^{(f)}(x) J_\mu^{(f')}(0)\rangle$$

with $\hat{x}_\nu = \min(x_\nu, L_\nu - x_\nu)$

Consistency of methods for determination of $\Pi(0)$

$\Pi(0)$ can be determined from fit to $\Pi(Q^2)$ (``traditional'') or from 2nd Fourier derivative (``2nd moment'' and ``Taylor'')

a (fm)	T (fm)	L (fm)	M_π (MeV)	$M_{K\chi}$ (MeV)	$M_\pi T$	$M_\pi L$
0.064	144a = 9.2	96a = 6.1	$1.32973(76)10^2$	$4.7929(20)10^2$	6.2	4.1



Similar consistency observed for s and c contributions

Strategy for determining $a_\mu^{\mathrm{HVP,LO}}$

Treat low-Q^2 and high-$Q^2 > 0.2\,\mathrm{GeV}^2$ separately [Golterman '12-14], flavor by flavor:

- Use one of the 3 methods for $Q^2 \leq 0.2\mathrm{GeV}^2$ (``traditional'', ``2nd moment'', ``Taylor'')
  $\rightarrow$ all 3 methods give Taylor coefficients $\Pi_{0,1,2}^{(ff)}$ simulation per simulation for each flavor f
  $\rightarrow$ interpolate $\Pi_{1,2}^{(ff)}$ to physical mass point and extrapolate to $a \rightarrow 0$
  $\rightarrow$ integrate low-order Padé description of $\hat{\Pi}^{(ff)}(Q^2)$ given by these physical $\Pi_{1,2}^{(ff)}$ to get $Q^2 \leq 0.2\,\mathrm{GeV}^2$ contributions to $a_\mu^{\mathrm{HVP,LO}}$
- Direct numerical integration (trapezoid) of $\hat{\Pi}^{(ff)}(Q^2)$ for $Q^2 > 0.2\mathrm{GeV}^2$

$\Rightarrow$ important low-Q^2 region not biased by more statistically precise high-Q^2 results

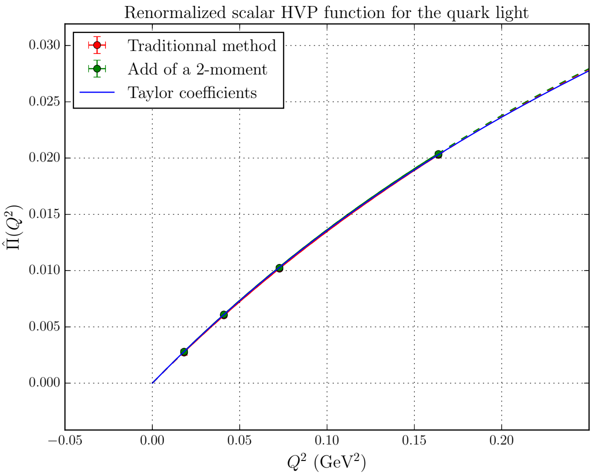
$\Rightarrow$ rational approximation at low Q^2 converges to true function [Golterman '12-14]

$\Rightarrow$ Padé $[1,1]$ for ud and $Q^2 \leq 0.2\mathrm{GeV}^2$ sufficient for 1% determination of $a_\mu^{\mathrm{HVP,LO}}$ [Golterman '12-14]

- Alternative (not tried): ``moments'' approximation based on Mellin transforms [de Rafael '14]

Comparison of 3 methods for $\hat{\Pi}(Q^2)$ with $Q^2 \leq 0.2\,\mathrm{GeV}^2$ for ud channel

a (fm)	T (fm)	L (fm)	M_π (MeV)	M_{K^x} (MeV)	$M_\pi T$	$M_\pi L$
0.064	144a = 9.2	96a = 6.1	1.32973(76)10^2	4.7929(20)10^2	6.2	4.1



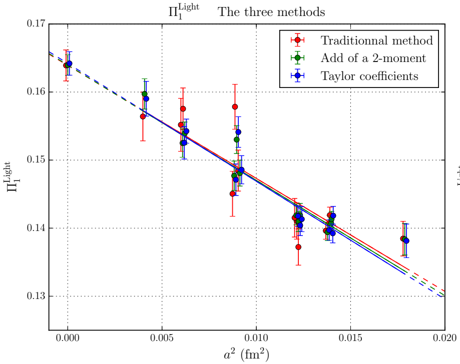
Physical fit strategy

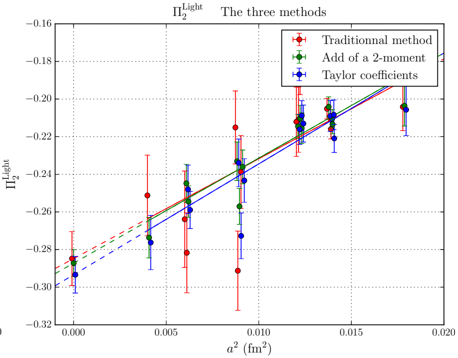
- Interpolate $\Pi_{1,2}^{(ff)}$ for each method to physical mass point and extrapolate to $a \to 0$

$$\Pi_{1,2}^{(ff)}(a_\beta, aM_\pi, aM_{K\chi}) = \Pi_{1,2}^{(ff),\phi}\Big\{ \quad 1 + p_l^{(ff)}\left[\left(\frac{aM_\pi}{a_\beta}\right)^2 - {M_\pi^\phi}^2\right] + p_s^{(ff)}\left[\left(\frac{aM_{K\chi}}{a_\beta}\right)^2 - {M_{K\chi}^\phi}^2\right] + \frac{p_{a_1}^{(ff)} a_\beta^2 + p_{a_2}^{(ff)} a_\beta^4}{1 + p_{b_1}^{(ff)} a_\beta^2}\Big\}$$

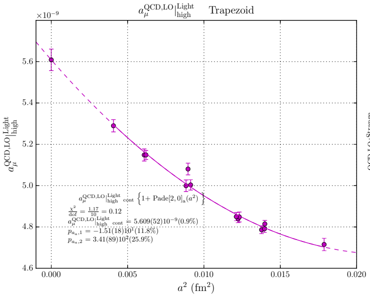
- Similarly for $Q^2 > 0.2\,\mathrm{GeV}^2$ contributions to $a_\mu^{\mathrm{HVP,LO}}$
- Statistical error obtained with resampling method (bootstrap)
- Systematic error associated with inter/extrapolation dominantly due to $a \to 0$
  - remove 1 or 2 of the 6 lattices spacings
  - central value is mean of distribution weighted by Akaike Information Criterion
  - error is square-root of variance of distribution

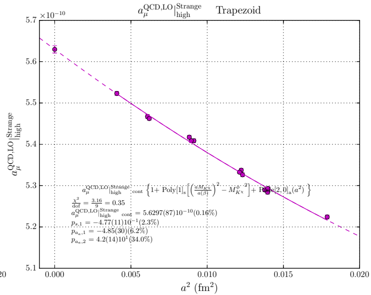
Physical fit of Π_{12}: light case

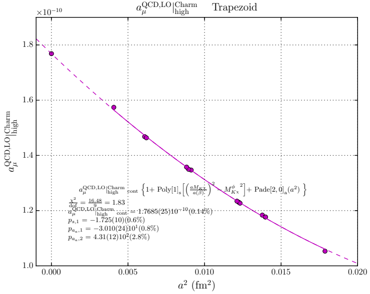




Physical fit of $a_\mu^{\mathrm{HVP,LO}}$ for $Q^2 > 0.2\,\mathrm{GeV}^2$







Final results for $\Pi_{1,2}^{(ff')}$

The Budapest-Marseille-Wuppertal collaboration recently computed (not part of my thesis):

- disconnected contributions to $a_\mu^{\rm HVP,LO}$
- FV corr. computed analytically with XPT as per Aubin et al '15

	Π_1 [GeV^{-2}]	Π_2 [GeV^{-4}]
Light	$1.642(17)(17)10^{-1}$	$-2.934(97)(82)10^{-1}$
Strange	$6.5682(63)(22)10^{-2}$	$-5.3109(90)(24)10^{-2}$
Charm	$4.0150(53)(121)10^{-3}$	$-2.581(13)(23)10^{-4}$
Disconnected	$-1.50(20)(10)10^{-2}$	$4.60(100)(40)10^{-2}$
Light FV corr.	$0.0011(45)$	$-3.2(21)10^{-2}$
I=1 FV corr.	= (Light FV corr.)/2	
total + I=1 FV corr.	$9.919(90)(84)(225)10^{-2}$	$-1.799(50)(41)(105)10^{-1}$

Isospin breaking and QED corrections not included: $1 \div 2\%$

Comparison with phenomenology

Π_1 agrees with the only other published value, based on $e^+e^- \to \text{hadrons}$ data and dispersion relations : [Benayoun et al '16]

$$\Pi_1^{\text{BMW,preliminary}} = 0.0992(9)(8)(22)\,\text{GeV}^{-2}$$
$$\Pi_1^{\text{pheno}} = 0.0990(7)\,\text{GeV}^{-2}$$

and slight disagreement for Π_2 :

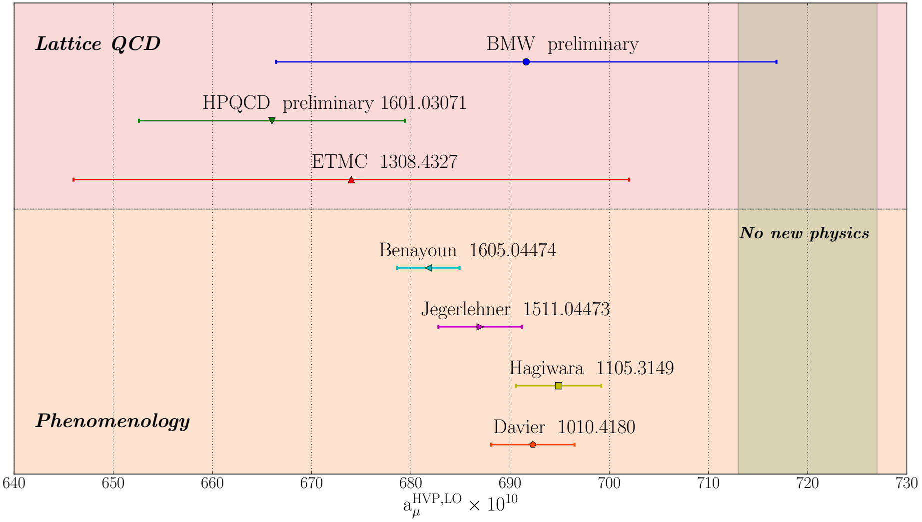
$$\Pi_2^{\text{BMW,preliminary}} = -0.180(5)(4)(10)\,\text{GeV}^{-4}$$
$$\Pi_2^{\text{pheno}} = -0.2057(16)\,\text{GeV}^{-4}$$

Final results for contributions to $a_\mu^{\mathrm{HVP,LO}}$

	$[0, 0.2\ \mathrm{GeV}^2]$	$[0.2, \infty\ \mathrm{GeV}^2]$	Total flavor
Light	$578(22)10^{-10}$	$56.09(91)10^{-10}$	$634(23)10^{-10}$
Strange	$47.935(54)10^{-10}$	$5.630(10)10^{-10}$	$53.565(64)10^{-10}$
Charm	$12.080(41)10^{-10}$	$1.7685(26)10^{-10}$	$13.849(43)10^{-10}$
Disconnected	$-10.1(19)10^{-10}$	$-$	$-10.1(19)10^{-10}$
Total range	$628(24)10^{-10}$	$63.49(93)10^{-10}$	$692(25)10^{-10}(3.6\%)$

Isospin breaking and QED corrections not included: $1 \div 2\%$

Lattice vs phenomenology



Conclusion and future prospects

Summary

- $a_\mu^{\mathrm{HVP,LO}}$ computed in Lattice QCD with physical Pion/Kaon masses
- $N_f = (2+1+1)$: two degenerate light quarks + strange + charm
- Space-time boxes : $T > L \gtrsim 6$ fm
- 6 lattice spacing : $a = 0.134 \searrow 0.064$ fm
- Statistics + Systematics : $a_\mu^{\mathrm{HVP,LO}} = (692 \times 10^{-10}) \pm 3\%$

Future prospects

- Full understanding of FV corrections
- Improve statistics
- Compute with Isospin breaking / QED effects (1, 2 % effect)
- Compute a_μ^{HLbL}
- Fermilab E989 start in April 2017 !
- Theory has to follow : $\frac{\delta a_\mu^{\mathrm{QCD}}}{a_\mu^{\mathrm{QCD}}} \searrow 0.2\%$

Articles

- Sz. Borsanyi, Z. Fodor, T. Kawanai, S. Krieg, L. Lellouch, R. Malak, K. Miura, K. Szabo, C. Torrero, B. Toth, arXiv:1612.02364 [hep-lat].
- R. Malak, Z. Fodor, C. Hoelbling, L. Lellouch, A. Sastre and K. Szabo, arXiv:1502.02172 [hep-lat].

→ Proceedings of the 32nd International Symposium on Lattice Field Theory (*Lattice 2014*)

- E. B. Gregory, Z. Fodor, C. Hoelbling, S. Krieg, L. Lellouch, R. Malak, C. McNeile and K. Szabo, arXiv:1311.4446 [hep-lat].

→ Proceedings of the 31st International Symposium on Lattice Field Theory (*Lattice 2013*)

- S. Dürr, Z. Fodor, C. Hoelbling, S. Krieg, T. Kurth, L. Lellouch, T. Lippert and R. Malak *et* al., arXiv:1310.3626 [hep-lat].

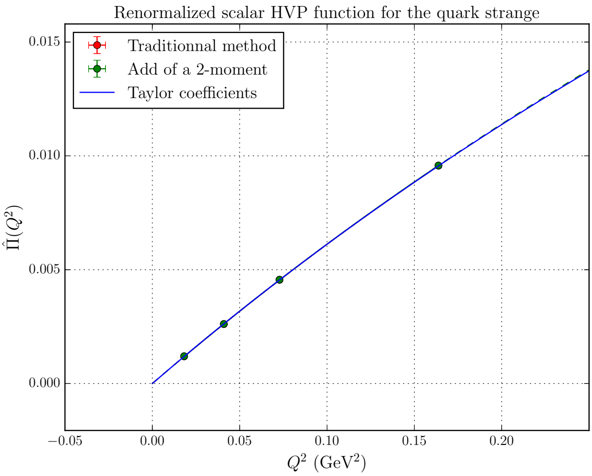
→ Phys. Rev. D 90, 114504 (2014)

Thank you for your attention !

$\hat{\Pi}(Q^2)$ @ low Q^2: comparison of 3 methods (strange)

Comparison of 3 methods for $\hat{\Pi}(Q^2)$ with $Q^2 \leq 0.2\,\mathrm{GeV}^2$ for s channel

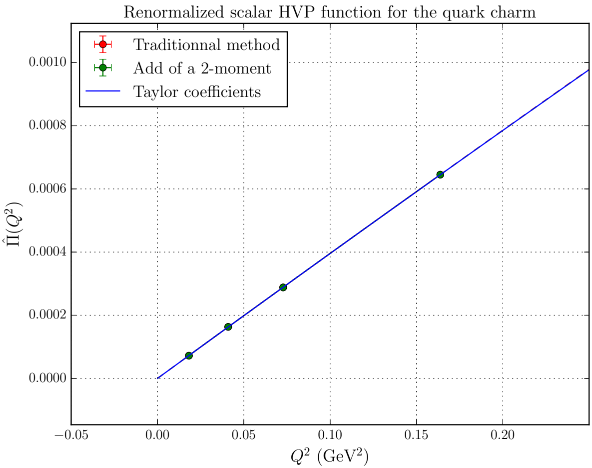
a (fm)	T (fm)	L (fm)	M_π (MeV)	M_{K^X} (MeV)	$M_\pi T$	$M_\pi L$
0.064	144a = 9.2	96a = 6.1	$1.32973(76)10^2$	$4.7929(20)10^2$	6.2	4.1



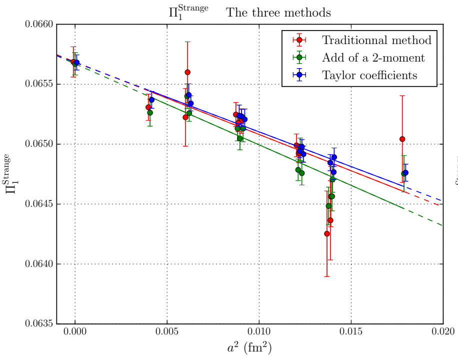
$\hat{\Pi}(Q^2)$ @ low Q^2: comparison of 3 methods (charm)

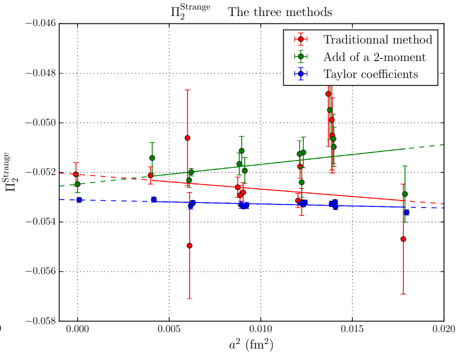
Comparison of 3 methods for $\hat{\Pi}(Q^2)$ with $Q^2 \leq 0.2\,\mathrm{GeV}^2$ for c channel

a (fm)	T (fm)	L (fm)	M_π (MeV)	M_{K^χ} (MeV)	$M_\pi T$	$M_\pi L$
0.064	144a = 9.2	96a = 6.1	$1.32973(76)10^2$	$4.7929(20)10^2$	6.2	4.1

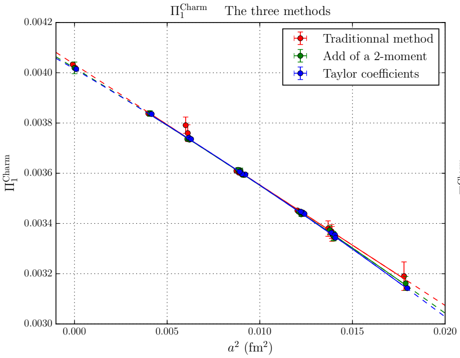


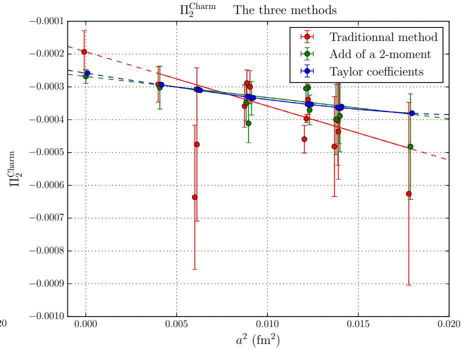
Physical fit of Π_{12}: strange case





Physical fit of Π_{12}: charm case





Dispersion relation